2. Krasovskii, N. N., Game Problems on the Contact of Motions. Moscow, "Nauka", 1970.
3. Krasovskii, N. N., Game problems of dynamics. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, №5, 1969.
4. Krasovskii, N. N. and Subbotin, A. I., An alternative for the game problem of convergence. PMM Vol. 34, №6, 1970.
5. Fedorenko, R. P., On one class of differential pursuit games. J. USSR Comput. Math. and Math. Phys., Pergamon Press, Vol. 10, №5, 1970.
6. McKinsey, J. C. C., Introduction to the Theory of Games. New York, McGraw-Hill Book Co., Inc., 1952.
7. Kolmogorov, A. N. and Fomin, S. V., Elements of the Theory of Functions and Functional Analysis. Moscow, "Nauka", 1968.

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ON INVESTIGATING THE STABILITY OF NEARLY-CRITICAL SYSTEMS

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In [1 - 3] the proof was given of the reduction principle in stability theory when investigating critical cases. In the present paper the reduction principle is proved for nearly-critical cases [4]. The stability problem in one essentially singular case is solved. The stability of a pitch gyro is investigated.

1. We consider a real autonomous system of differential equations of a perturbed motion of the form

$$x_\nu^{\bullet} = \sum_{l=1}^{r} a_{\nu l} x_l + X_\nu(x) \qquad (\nu = 1, \ldots, r, \; x \equiv x_1, \ldots, x_r) \tag{1.1}$$

Here X_ν are holomorphic functions in the region

$$x_1^2 + \ldots + x_r^2 \leqslant H \tag{1.2}$$

whose expansions do not contain terms of less than second order. H is some finite positive number. We assume that the characteristic equation of system (1.1) has q roots with negative real parts, m zero roots, and p roots with real parts which are small in absolute value. We remark that any system with an arbitrary number of zero and pure-imaginary roots and roots with small positive real parts can be reduced to such a form.

Under these conditions system (1.1) can be transformed by means of linear substitutions to the form

$$\begin{aligned} y_s^{\bullet} &= \sum_{k=1}^{n} g_{sk} y_k + Y_s(y, z) \qquad & \delta = k_1 + \ldots + k_n, \quad s = 1, \ldots, n \\ & & j = 1, \ldots, q, \quad n = m + p \\ z_j^{\bullet} &= \sum_{i=1}^{q} p_{ji} z_i + \sum_{\delta \geqslant 2}^{\infty} A_j^{(*)} y^k + Z_j(y, z) \qquad & n + q = r, \quad y^k = y_1^{k_1} \ldots y_n^{k_n} \end{aligned} \tag{1.3}$$

Here y_s are critical and nearly-critical variables, z_j are the variables of the adjoint system. Here and subsequently the sign (*) replaces the index $k_1, \ldots, k_n$. The stability or instability of system (1.1) will be defined in the following way [4]: if in the space of $x_1, \ldots, x_r$ we can find a closed region G possessing the property that the perturbations $x_1, \ldots, x_r$, considered as functions of time and satisfying the equations of perturbed motion, do not leave this region for any value of $t \geqslant t_0$ provided their initial values are located inside or on the boundary of this region, then the unperturbed motion is stable; otherwise, it is unstable.

The problem under consideration subsequently is reduced to the obtaining of necessary and sufficient conditions imposed on the right-hand sides of the system of equations of perturbed motion. This is done in order that the stability region G, in spite of the presence in the characteristic equation of roots with positive real parts, be located in a sufficiently small neighborhood of the origin, and that all perturbed motions starting in region G approximate the unperturbed motion in time. To solve this problem it is necessary to establish for system (1.1) the existence of the region G, assuming that the number H in (1.2) is sufficiently small, and to indicate roughly how the boundaries of the stability region are determined. Otherwise it is necessary to show that in a sufficiently small neighborhood of the origin there does not exist a stability region with the properties indicated, and the motion is unstable. This problem is of interest in applications and in a general formulation answers the question on the "dangerous" and "safe" boundaries of the stability region [5].

Obviously, region G is a stability region with the above-mentioned properties if in it there exists a sign-definite Liapunov function V [6] with a sign-definite derivative of opposite sign; conversely, there is no region G if in (1.2) there exists a Chetaev function [7]. We assume that the signs of the derivatives of these functions are determined by forms of no higher than Nth order, independently of higher-order forms.

Let us show that the presence or absence of a region G in a sufficiently small neighborhood of the origin may be established from a "truncated" system, i.e., a system with only critical or nearly-critical variables, obtained from system (1.3) by means of known transformations [8]. By carrying out these transformations under the condition that the positive real parts of the roots of the characteristic equation are sufficiently small, in the place of system (1.3) we obtain

$$\eta_s^{\cdot} = \sum_{k=1}^{n} g_{sk}\eta_k + \sum_{\delta\geqslant 2}^{\infty} B_s^{(*)}\eta^k + \sum_{\delta\geqslant N+1}^{\infty} P_s^{(*)}(\zeta)\,\eta^k + H_s(\eta,\zeta) \qquad (1.4)$$

$$\zeta_j^{\cdot} = \sum_{i=1}^{q} p_{ji}\zeta_i + \sum_{\delta\geqslant N+1}^{\infty} A_{j1}^{(*)}\eta^k + \sum_{\delta\geqslant 1}^{\infty} Q_j^{(*)}(\zeta)\,\eta^k + E_j(\eta,\zeta)$$

Here the functions H_s and E_i vanish when $\zeta_1 = \ldots = \zeta_q = 0$ and do not contain linear terms in these variables, $B_s^{(*)}$, $A_{j1}^{(*)}$ are constant coefficients, $P_s^{(*)}$ and $Q_j^{(*)}$ are linear forms in $\zeta_1, \ldots, \zeta_q$.

Theorem. If for the truncated system

$$\eta_s^{\cdot} = \sum_{k=1}^{n} q_{sk}\eta_k + \sum_{\delta\geqslant 2}^{N} B_s^{(*)}\eta^k \qquad (s = 1, \ldots, n) \qquad (1.5)$$

a stability region exists in a sufficiently small neighborhood of the origin, or, conversely it is unstable, and this has been established by means of a Liapunov or a Chetaev function the sign of whose derivative is determined by a form of no higher than Nth order independently of higher-order forms, then a stability region exists also for the complete system, or the system is unstable.

Indeed, for the complete system the function

$$V = V_1(\eta) + V_2(\zeta) \tag{1.6}$$

is a Liapunov or a Chetaev function. Here V_1 is a Liapunov or a Chetaev function for system (1.5), while V_2 is determined from the equation

$$\sum_{j=1}^{q} \frac{\partial V_2}{\partial \zeta_j}(p_{j1}\zeta_1 + \ldots + p_{jq}\zeta_q) = M(\zeta_1^2 + \ldots + \zeta_q^2) \tag{1.7}$$

The number $M < 0$ for a positive-definite function V_2 with a derivative which is negative-definite in region G The derivative of function (1.6) by virtue of system (1.4) is

$$V' = V_1'(\eta) + M(\zeta_1^2 + \ldots + \zeta_q^2) + \sum_{s=1}^{n} \frac{\partial V_1}{\partial \eta_s}\Big[\sum_{\delta \geqslant N+1}^{\infty} B_s^{(*)}\eta^k +$$

$$+ \sum_{\delta \geqslant N+1}^{\infty} P_s^{(*)}(\zeta)\eta^k + H_s(\eta, \zeta)\Big] + \sum_{j=1}^{q} \frac{\partial V_2}{\partial \zeta_j}\Big[\sum_{\delta \geqslant N+1}^{\infty} A_{j1}^{(*)}\eta^k +$$

$$+ \sum_{\delta \geqslant 1}^{\infty} Q_j^{(*)}(\zeta)\eta^k + E_j(\eta, \zeta)\Big] \tag{1.8}$$

This expression can be written in the form

$$V' = V_1' + M(\zeta_1^2 + \ldots + \zeta_q^2) + \sum_{\delta \geqslant N+1}^{\infty} R^{(*)}(\zeta)\eta^k + \sum_{i=1}^{q}\sum_{j=1}^{q} \zeta_i\zeta_j F_{ij}(\eta, \zeta) \tag{1.9}$$

In the region of changing of the variables being considered, the sign of V' is determined by the first two terms independently of the terms with the functions $R^{(*)}$ and F_{ij}. Consequently, a stability region exists also for the complete system.

Let us now assume that V_1 is a Chetaev function for system (1.5). Then either the region $V_1 > 0$ is contained inside the region $V_1' > 0$ or in the region $V_1 V_1' > 0$ we can delineate a region where a certain function $W \geqslant 0$ and the values of W' are of the same sign on the boundary $(W = 0)$. If we assume that the region $V_1 > 0$ is contained inside the region $V_1' > 0$, we determine the function V_2 from (1.7) with $M > 0$. The derivative of the function V relative to Eqs. (1.4) also can be represented in form (1.9). It is also necessary to take into account that the functions F_{ij} need not vanish when all the variables are zero. Let us define the number $M > 0$, so that the function

$$L(\zeta) = M(\zeta_1^2 + \ldots + \zeta_q^2) + \sum_{i=1}^{q}\sum_{j=1}^{q} \zeta_i\zeta_j F_{ij}^{(0)}$$

is a positive-definite quadratic form for $F_{ij}^{(0)} = F_{ij}\ (0,\ 0)$. By representing $F_{ij} = F_{ij}^{(0)} + F_{ij}^{(1)}\ (\zeta, \eta)$, we write expression (1.9) in the form

$$V' = V_1'(\eta) + L(\zeta) + \sum_{\delta \geqslant N+1}^{\infty} R^{(*)}(\zeta)\eta^k + \Sigma\Sigma\, \zeta_i\zeta_j F_{ij}^{(1)}(\zeta, \eta)$$

For sufficiently small η_s, ζ_j the last sums do not alter the sign of V' as determined by the first two terms. Consequently, when $V > 0$ we have $V' > 0$ because the function $'V_2 < 0$, and V is a Chetaev function for the complete system. The proof for the functions V and W satisfying Chetaev's theorem can be carried out analogously. A method has been presented in [9] for an approximate estimate of the boundary of stability region G for a system containing pure-imaginary roots and complex roots with small positive real parts.

2. As was shown in Sect. 1 we can make conclusions on the existence of region G on the basis of the truncated system if and only if the problem is solved by forms of a finite order N As is known, the latter holds in the case when the series of u_j determined from the system

$$\sum_{s=1}^{n} \frac{\partial u_j}{\partial y_s}\Big[\sum_{k=1}^{n} g_{sk}y_k + Y_s(y, u)\Big] = \sum_{i=1}^{q} p_{ji}u_i + \Sigma A_j^{(*)}y^k + Z_j(y, u) \quad (2.1)$$

do not make the expressions for $Y_s(y, u)$ vanish identically. When $Y_s(y, u) \equiv 0$ forms of any finite order do not solve the stability problem, and it is necessary to examine the complete system (1.3).

Let $Y_s(y, u) \equiv 0$; we consider this essentially singular case by assuming that the equation $| g_{sk} - \delta_{sk}\nu | = 0$ either has no multiple roots or, in the presence of multiple roots, to each such roots there corresponds as many groups of solutions as its multiplicity. We transform system (1.3) by the substitution

$$z_j = \zeta_j + u_j(y) \quad (2.2)$$

Here the u_j form series satisfying the system of Eqs. (2.1). Under our assumptions concerning the roots ν_s and for $Y_s(y, u) \equiv 0$ the series of u_j converge absolutely, on the basis of Kamenkov's theorem [8], in the region being considered of changing of the variables. As a result of transformation (2.2) the system of equations (1.3) takes the form

$$y_s^{\cdot} = \sum_{k=1}^{n} g_{sk}y_k + \sum_{j=1}^{q} Y_{sj}(y)\zeta_j + Y_s(y, \zeta) \quad (2.3)$$

$$\zeta_j^{\cdot} = \sum_{i=1}^{q} p_{ji}\zeta_i + \sum_{i=1}^{q} Z_{ji}(y)\zeta_i + Z_j^1(y, \zeta)$$

Here the functions Y_{sj} and Z_{ji} vanish when $y_1 = \ldots = y_n = 0$, while the functions Y_s and Z_j^1 do not contain linear terms in the ζ_j. By transforming system (2.3) to canonical form, we obtain

$$\xi_s^{\cdot} = \mu_s\xi_s - \lambda_s\eta_s + \sum_{j=1}^{q} \Xi_{sj}(\xi, \eta, r, \rho)\zeta_j + \Xi_s(\xi, \eta, r, \rho, \zeta)$$

$$\eta_s^{\cdot} = \mu_s\eta_s + \lambda_s\xi_s + \sum_{j=1}^{q} H_{sj}(\xi, \eta, r, \rho)\zeta_j + H_s(\xi, \eta, r, \rho, \zeta)$$

$$r_k^{\cdot} = \mu_{k1}r_k + \sum_{j=1}^{q} R_{kj}(\xi, \eta, r, \rho)\zeta_j + R_k(\xi, \eta, r, \rho, \zeta) \quad (2.4)$$

$$\rho_\mu^{\cdot} = \sum_{j=1}^{q} S_{\mu j}(\xi, \eta, r, \rho)\zeta_j + S_\mu(\xi, \eta, r, \rho, \zeta)$$

$$\zeta_j^{\cdot} = \sum_{i=1}^{q} p_{ji}\zeta_i + \sum_{i=1}^{q} Q_{ji}(\xi, \eta, r, \rho)\zeta_i + Z_j(\xi, \eta, r, \rho, \zeta)$$

$(s = 1, \ldots, l,\ k = 1, \ldots, g,\ 2l + g = p,\ \mu = 1, \ldots, m,\ j = 1, \ldots, q,\ \mu_s > 0,\ \mu_{k1} > 0)$

We introduce new variable x_s, y_s, ν_k, χ_μ in the following way:

$$\xi_s = x_s + \Sigma\zeta_j u_{sj}, \qquad \eta_s = y_s + \Sigma\zeta_j v_{sj} \qquad (2.5)$$
$$r_k = \nu_k + \Sigma\zeta_j w_{kj}, \qquad \rho_\mu = \chi_\mu + \Sigma\zeta_j f_{\mu j}$$

and we determine the functions u_{sj}, v_{sj}, w_{kj}, $f_{\mu j}$ from the equations

$$\sum_{\nu=1}^{l}\left[\frac{\partial F_\alpha}{\partial\xi_\nu}(\mu_\nu\xi_\nu - \lambda_\nu\eta_\nu) + \frac{\partial F_\alpha}{\partial\eta_\nu}(\mu_\nu\eta_\nu + \lambda_\nu\xi_\nu)\right] + \sum_{\delta=1}^{g}\frac{\partial F_\alpha}{\partial r_\delta}\mu_{\delta 1}r_\delta = F_{\alpha\alpha}$$
$$(\alpha = 1, 2, 3, 4)$$

$$F_1 = u_{sj},\ F_{11} = -\sum_{i=1}^{q} u_{si}p_{ij} - \lambda_s v_{sj} + \mu_s u_{sj} + \Xi_{sj} - \Sigma u_{si}Q_{ij}$$
$$F_2 = v_{sj},\ F_{22} = -\Sigma v_{si}p_{ij} + \lambda_s u_{sj} + \mu_s v_{sj} + \mathrm{H}_{sj} - \Sigma v_{si}Q_{ij} \qquad (2.6)$$
$$F_3 = w_{kj},\ F_{33} = -\Sigma w_{ki}p_{ij} + \mu_{k1}w_{kj} + R_{kj} - \Sigma w_{ki}Q_{ij}$$
$$F_4 = f_{\mu j},\ F_{44} = -\Sigma f_{\mu i}p_{ij} + S_{\mu j} - \Sigma f_{\mu i}Q_{ij}$$

On the basis of that same theorem in [8] the functions u_{sj}, v_{sj}, w_{kj}, $f_{\mu j}$ are determined from system (2.6) in the form of absolutely convergent series. As a result of transformation (2.5) system (2.4) takes the form

$$x_s^{\cdot} = \mu_s x_s - \lambda_s y_s + X_s(x, y, \nu, \chi, \zeta),\ y_s^{\cdot} = \mu_s y_s + \lambda_s x_s + Y_s(x, y, \nu, \chi, \zeta)$$
$$\nu_k^{\cdot} = \mu_{k1}\nu_k + P_k(x, y, \nu, \chi, \zeta),\ \chi_\mu^{\cdot} = \theta_\mu(x, y, \nu, \chi, \zeta) \qquad (2.7)$$
$$\zeta_j^{\cdot} = \Sigma p_{ji}\zeta_i + Z_j^1(x, y, \nu, \chi, \zeta)$$

Here X_s, Y_s, P_k, θ_μ, Z_j^1 vanish when $\zeta_1 = \ldots = \zeta_q = 0$, while, furthermore, the first four functions do not contain linear terms in the ζ_j.

For system (2.7) we take the Chetaev function in the form

$$V = {}^1/_2\Sigma(x_s^2 + y_s^2) + {}^1/_2\Sigma\nu_k^2 + \Sigma\chi_\mu^2 + W(\zeta) \qquad (2.8)$$

by computing the negative-definite quadratic form W in the variables ζ_j from the equation

$$\Sigma\frac{\partial W}{\partial\zeta_j}(p_{j1}\zeta_1 + \ldots + p_{jq}\zeta_q) = \zeta_1^2 + \ldots + \zeta_q^2$$

Then the derivative V' by virtue of system (2.7) can be represented as

$$V' = \Sigma\mu_s(x_s^2 + y_s^2) + \Sigma\mu_{k1}\nu_k^2 + \Sigma\zeta_j^2 + \Sigma\Sigma\zeta_i\zeta_j F_{ij}$$

Here the functions F_{ij} vanish when $x_s = y_s = \nu_k = \chi_\mu = \zeta_j = 0$. In the region $\Sigma\chi_\mu^2 \leqslant |W|$, $V > 0$ the function $V' > 0$ for values of the variables satisfying the inequality

$$|\Sigma\Sigma\zeta_i\zeta_j F_{ij}| < \Sigma\zeta_j^2$$

Thus, for a system (1.1) satisfying the conditions in Sect. 2, region G does not exist in a sufficiently small neighborhood of the origin, and the motion is unstable.

3. Example. Using the results of [9], we investigate the stability of the motion of a pitch gyro with a center of gravity which is displaced relative to the point of suspension, without regard to the Earth's rotation. We assume that the correction of the position of the gyro output axis relative to the direction of the local vertical is effected by a pendulum correcting device as well as owing to the moment arising from the displacement of the center of gravity. The equations of the motion of the gyro output axis relative to the local vertical can be represented in the form (the notation has the same meaning as in [10]) (*)

$$J_{\text{в}} v^{\cdot\cdot} + J\Omega\psi^{\cdot} = -Glv + M_{\text{кв}}, \qquad J_c\psi^{\cdot\cdot} - J\Omega v^{\cdot} = -Gl\psi + M_{\text{кс}} \tag{3.1}$$

$$J_1\varepsilon_1^{\cdot\cdot} + \varkappa_1\varepsilon_1^{\cdot} + k_1\varepsilon_1 = M_{g1}(\psi^{\cdot} - \varepsilon_1^{\cdot}), \qquad J_2\varepsilon_2^{\cdot\cdot} + \varkappa_2\varepsilon_2^{\cdot} + k_2\varepsilon_2 = M_{g2}(v^{\cdot} - \varepsilon_2^{\cdot})$$

When the gyro output axis deviates from the vertical the suspension axes of the pendulums rotate with angular velocities $\psi^{\cdot}$ and $v^{\cdot}$. By considering the system's motion only until the instant when $\psi^{\cdot}$ or $v^{\cdot}$ equal zero, we take each pendulum to be a Froude pendulum [11] with the characteristics

$$M_{g1}(\psi^{\cdot}) = l - l_1\psi^{\cdot}, \qquad M_{g2}(v^{\cdot}) = m - m_1 v^{\cdot}$$

Let us rewrite the equations for the oscillations of the pendulums as

$$J_1\varphi_1^{\cdot\cdot} - (l_1 - \varkappa_1)\varphi_1^{\cdot} + k_1\varphi_1 = -l_1\psi^{\cdot}, \qquad J_2\varphi_2^{\cdot\cdot} - (m_1 - \varkappa_2)\varphi_2^{\cdot} + k_2\varphi_2 = -m_1 v^{\cdot}$$

As in the case of a Froude pendulum we assume that

$$(l_1 - \varkappa_1)\,/\,J_1 = 2\mu_2 > 0, \qquad (m_1 - \varkappa_2)\,/\,J_2 = 2\mu_3 > 0$$

where μ_2 and μ_3 are small positive numbers.

By approximating $M_{\text{кв}}$ and $M_{\text{кс}}$ as [12]

$$M_{\text{кв}} = -[q_1(\psi - \varphi_1)^3 + q_2(\psi - \varphi_1)^5 + \ldots], \qquad M_{\text{кс}} = h_1(v - \varphi_2)^3 + h_2(v - \varphi_2)^5 + \ldots$$

$$(q_1 > 0,\ h_1 > 0)$$

by introducing

$$k_1\,/\,J_1 = \lambda_2^2 + \mu_2^2,\ k_2/J_2 = \lambda_3^2 + \mu_3^2,\ l_1/J_1 = c_1,\ m_1/J_2 = c_2$$
$$Gl/J\Omega = \lambda_1,\ \psi = x_1,\ v = y_1,\ \varphi_1 = y_2,\ y_2^{\cdot} = \mu_2 y_2 + \lambda_2 x_2,\ \varphi_2 = y_3$$
$$y_3^{\cdot} = \mu_3 y_3 + \lambda_3 x_3,\ q_1/J\Omega = a_1,\ h_1/J\Omega = b_1,\ c_1\lambda_1/\lambda_2 = a_2$$
$$c_1 q_1/\lambda_2 J\Omega = b_2,\ c_2\lambda_1/\lambda_3 = a_3,\ c_2 h_1/\lambda_3 J\Omega = b_3$$

and by neglecting the nutation terms in (3.1), we obtain the system of equations

$$\begin{aligned} x_1^{\cdot} &= -\lambda_1 y_1 - a_1(x_1 - y_2)^3 - \ldots, \quad y_1^{\cdot} = \lambda_1 x_1 - b_1(y_1 - y_3)^3 - \ldots \\ x_2^{\cdot} &= \mu_2 x_2 - \lambda_2 y_2 + a_2 y_1 + b_2(x_1 - y_2)^3 + \ldots, \quad y_2^{\cdot} = \mu_2 y_2 + \lambda_2 x_2 \\ x_3^{\cdot} &= \mu_3 x_3 - \lambda_3 y_3 - a_3 x_1 + b_3(y_1 - y_3)^3 + \ldots, \quad y_3^{\cdot} = \mu_3 y_3 + \lambda_3 x_3 \end{aligned} \tag{3.2}$$

The characteristic equation corresponding to system (3.2) has two pairs of complex-conjugate roots with small positive real parts and a pair of pure-imaginary roots $\pm i\lambda_1$. By the change of variables

$$x_1 = \xi_1,\ y_1 = \eta_1,\ x_2 = \xi_2 + \alpha_1\eta_1 + \beta_1\xi_1,\ y_2 = \eta_2 + \alpha_2\eta_1 + \beta_2\xi_1$$
$$x_3 = \xi_3 + \alpha_3\xi_1 + \beta_3\eta_1,\ y_3 = \eta_3 + \alpha_4\xi_1 + \beta_4\eta_1$$

(*) Editorial Note. Reference [10] is unobtainable. However the deduction made from the Russian original results in that the subscripts denote the following: к stands for "correction"; в for the "vertical"; and с for "displacement".

we transform system (3.2) to the canonical form

$$\xi_s^{\cdot} = \mu_s\xi_s - \lambda_s\eta_s + \Xi_s(\xi, \eta), \qquad \eta_s^{\cdot} = \mu_s\eta_s + \lambda_s\xi_s + H_s(\xi, \eta)$$

$$\Xi_1 = -a_1X_1 + \ldots, H_1 = -b_1Y_1 + \ldots, \qquad X_1 = (\xi_1 - \eta_2 - \alpha_2\eta_1 - \beta_2\xi_1)^3 \qquad (3.3)$$
$$Y_1 = (\eta_1 - \eta_3 - \alpha_4\xi_1 - \beta_4\eta_1)^3, \qquad \Xi_2 = (b_2 + \beta_1a_1)X_1 + \alpha_1b_1Y_1 + \ldots$$
$$H_2 = \alpha_2b_1Y_1 + \beta_2a_1X_1 + \ldots, \qquad \Xi_3 = \alpha_3a_1X_1 + (b_3 + \beta_3b_1)Y_1 + \ldots$$
$$H_3 = \alpha_4a_1X_1 + \beta_4b_1Y_1 + \ldots \qquad (s = 1, 2, 3, \mu_1 = 0)$$

The coefficients $\alpha_1, \ldots, \beta_4$ can be expressed without difficulty in terms of the coefficients of system (3.2). Note that

$$\beta_2 = -2a_2\lambda_1\lambda_2\mu_2/\Delta, \qquad \beta_4 = -2a_3\lambda_1\lambda_3\mu_3/\Delta \qquad (3.4)$$
$$\Delta = -[(\lambda_1^2 - \lambda_2^2)^2 + 2\mu_2^2(\lambda_1^2 + \lambda_2^2) + \mu_2^4]$$

The coefficients β_2 and β_4 are positive; it is easy to choose them less than unity. By making further transformations analogous to those made in [9], we obtain the system

$$r_s^{\cdot} = \mu_sr_s + r_s(a_{s1}r_1^2 + a_{s2}r_2^2 + a_{s3}r_3^2) + \ldots \quad (r_s \geqslant 0,\ s = 1, 2, 3,\ \mu_1 = 0) \qquad (3.5)$$

$$a_{11} = -{}^3/_8\{a_1(1 - \beta_2)[(1 - \beta_2)^2 + \alpha_2^2] + b_1(1 - \beta_4)[\alpha_4^2 + (1 - \beta_4)^2]\} < 0$$
$$a_{12} = -{}^3/_4a_1(1 - \beta_2) < 0, \quad a_{13} = -{}^3/_4b_1(1 - \beta_4) < 0$$
$$a_{21} = -{}^3/_4[\beta_2a_1(1 - \beta_2)^2 + \beta_2a_1\alpha_2^2] < 0, \qquad a_{22} = -{}^3/_8a_1\beta_2 < 0, \qquad a_{23} = 0$$
$$a_{31} = -{}^3/_4[\beta_4b_1(1 - \beta_4)^2 + \beta_4b_1\alpha_4^2] < 0, \qquad a_{33} = -{}^3/_8b_1\beta_4 < 0, \qquad a_{32} = 0,$$

From the first equation of system (3.5) it follows that asymptotic stability holds with respect to the coordinate r_1 and, consequently, with respect to the coordinates ψ and v. Instability regions exist with respect to the coordinates x_2, y_2 and x_3, y_3 The outer boundaries of these regions can be estimated approximately by the equalities

$$r_{20} = \sqrt{-\mu_2/a_{22}}, \qquad r_{30} = \sqrt{-\mu_3/a_{33}}$$

The values of r_{20} and r_{30} may be made sufficiently small by an appropriate choice of the system parameters. Simultaneously, the values of r_{20}, r_{30} determine approximately also the inner boundary of the stability region with respect to r_2, r_3. To estimate the outer boundary of the stability region, when it is possible to do so by finite order terms, it is necessary to write down the higher-order terms in the system (3.5). Note that from the instant that the gyro wheel axis achieves the equilibrium position, the equations of oscillations of the pendulums have positive coefficients for the first derivatives, and their motion is asymptotically stable.

BIBLIOGRAPHY

1. Kamenkov, G.V., On the stability of motion in a singular case. Tr. Kazansk. Aviats. Inst., №4, 1935.
2. Malkin, I.G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
3. Pliss, V.A., The principle of reduction in the stability theory of motion. Izv. Akad. Nauk SSSR, Ser. Mat. Vol.28, №6, 1964.
4. Kamenkov, G.V., On the stability of motion in nearly-critical cases. Tr. Univ. Druzhby Narodov im. P. Lumumby, Ser. Teor. Mekhan. Vol.1, №1, 1963.
5. Bautin, N.N., Behavior of Dynamic Systems Near the Boundaries of Stability Regions. Moscow-Leningrad, Gostekhizdat, 1950.

6. Liapunov, A. M., Collected Works, Vol. 2, General Problems of the Stability of Motion. Moscow, Izd. Akad. Nauk SSSR, 1956.
7. Chetaev, N. G., Stability of Motion. Moscow, Gostekhizdat, 1955.
8. Kamenkov, G. V., The stability of periodic motions. PMM Vol. 31, №1, 1967.
9. Veretennikov, V. G., Regions of stability in a case close to the critical one. PMM Vol. 35, №1, 1971.
10. Pavlov, V. A., Theory of Gyroscope and of Gyroscopic Devices. Leningrad, "Sudostroenie", 1964.
11. Andronov, A. A., Vitt, A. A., and Khaikin, S. E., Theory of Oscillations. Moscow, Fizmatgiz, 1959.
12. Veretennikov, V. G., On the stabilization of neutral systems. Tr. Univ. Druzhby Narodov im.P. Lumumby, Ser. Teor. Mekhan. Vol. 27, №5, 1968.

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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF CERTAIN NONLINEAR PROBLEMS OF HYDRODYNAMICS

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Periodic solutions of heat conduction equations with boundary conditions of the relay kind are found for a finite interval, and the behavior of such solutions at unlimited time increase is analyzed. Periodic solutions of heat conduction equations with nonlinear boundary conditions were considered in [1 - 4, 10], while in [5, 6] periodic solutions of nonhomogeneous heat conduction equations with their right-hand sides nonlinear with respect to the unknown functions are presented, and the asymptotic behavior of related initial problems is analyzed. Solutions of this kind define self-oscillating processes occurring in various branches of hydrodynamics (theory of filtration and diffusion [3 - 6]).

1. The problem reduces to finding the periodic solution of equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

in the finite region $-l < x < 0$ with boundary conditions

$$\frac{\partial u(-l,t)}{\partial x} = \begin{cases} h_1 u(-l,t) + q_1 & \text{for} \quad u(-l,t) < u_* \\ h_2 u(-l,t) + q_2 & \text{for} \quad u(-l,t) > u_{**} \end{cases} \tag{1.2}$$
$$(u_* > u_{**},\ h_1 > 0,\ h_2 > 0,\ q_2 > q_1)$$
$$u(0,t) = 0$$

We set $u(-l, t) = u_*$ at $t = T_1$ and $u(-l, t) = u_{**}$ at $t = T$, with $u = u_1(x, t)$ for $0 \leqslant t \leqslant T_1$ and $u = u_2(x, t)$ for $T_1 \leqslant t \leqslant T$, and seek the solution of this problem in the form of series